Reply by Authors to R.P. Eddy

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THE authors appreciate Eddy's comments on Ref. 1. However we should comment on some of the points he raised. The statement made near the end of Ref. 1 to which Eddy's first sentence refers is: "Considered to be the main contribution here is a scheme that determines appropriate successive approximations for use in the Aitken/Shanks formula to accelerate the iterative convergence..." The scheme referred to is a new approach for finding successive approximations to vector solutions of partial differential equations in a *most nearly geometric sequence* for use in the Aitken acceleration formula. Because Shanks[2] has shown that the formula works best if the sequence is "nearly geometric," the approach described in Ref. 1 seeks to obtain successive iterates that are in a most nearly geometric sequence, by automatically producing an appropriate "artificial power series" at each field point. A modified form of the method has been given in Ref. 3, with additional results given in Ref. 4. The two forms of the acceleration technique are described in more detail in Ref. 5.

Eddy states that his main purpose is to emphasize the conditions under which the Aitken/Shanks formula can be used to extrapolate to the sum of a series. He then starts with his Eq. (2) as a condition and obtains the Aitken/Shanks formula (often called Aitken's Δ^2 process). Therefore, he has shown, as is well known,[2,6] that his Eq. (2) with constant "a" is a sufficient condition for the Aitken formula to give an exact extrapolation. However, it is by no means a necessary condition for the formula to be effective, as implied by Eddy's description. For example, Ref. 5 gives an example problem ("Example 2") for which the iterated solutions are "nearly geometric" (as defined by Shanks[2] in the limit of large n) with the ratio of terms (Eddy's "a") in the series being $u'_{n+1}/u'_n = -x(n+2)/(n+1)$; and at $x=0.5$ the solutions for the terms (u'_1, u'_2, u'_3) are (1.00, −.75, .50). Thus, the ratios of terms are respectively −¾ and −⅔ at the beginning of the series, and so that ratio is not nearly a constant for the first three terms. Nevertheless, the extrapolation formula applied to those three terms gives $u^*(.5) = .55$, which has only a one-percent error from the exact solution, $u(.5) = 5/9$. Furthermore, in the subsonic and slightly supercritical flow cases computed and listed in Table 2 of Ref. 1, the method also worked very well at the beginning of the sequence of iterations. Thus, for solutions that are expected to be sufficiently well-behaved it seems to be worthwhile to attempt to use the acceleration method near the beginning of the iteration, not just after many iterations, and not just when the ratio "a" is nearly independent of n. Thus, although we appreciate Eddy's derivation of the conditions, Eqs. (17) or (19), under which the method would "work perfectly" (if U were a scalar), we feel that the method should not have the restrictive conditions that Eddy would impose, but that it can also be used in situations where intuition and experience suggest it may be useful.

Of course, the method is not guaranteed to work well at the beginning of the sequence in all cases. For example, shock waves may occur that are not well-defined or well-located when the iteration starts, and then early use of the extrapolation may give a very poor result. However, U_0 may be the resulting solution from any previous iteration (e.g., at some n when the solution is nearly converged), and U_1, U_2, U_3 (where $u_n(x)$ is a component of the vector $U_n(x)$ are then succeeding iterates found according to the special procedure.

It should be mentioned that a difficulty can occur when the solution U_n is close to convergence at one or more points in the field. Destructive errors are introduced by the subtractions and the consequent loss of significant figures in the use of the form $(u_1u_3 - u_2{}^2)/(u_1 - 2u_2 + u_3)$ when successive iterates are nearly constant. Because of this, a modified version of the procedure was developed. The modification uses the different form (e.g., see Shanks[2])

$$u^*(x) = u'_1 - (u'_2)^2/(u'_3 - u'_2) \tag{1}$$

where

$$\begin{aligned} u'_n(x) &= u_n - u_{n-1},\ n = 2,3,\ldots, \\ &= u_1,\ n = 1 \end{aligned} \tag{2}$$

for the extrapolation. This method was used in both Refs. 3 and 4 to accelerate transonic-flow cases.

Finally, the authors were aware from reading Shanks[2] and others that the Aitken Δ^2 process is a simple Padé approximant if the successive iterates to be used are partial sums of a power series. And even though the elegant ϵ-algorithm of Wynn, with the longer sequences introduced by Shanks, could be used,[5] we have emphasized the use of the simplest extrapolation formula because it requires less computer storage than other forms of the ϵ-algorithm, and the eventual applications are expected to be those numerical problems requiring significant computer storage.

References

[1] Martin, E.D and Lomax, H., "Rapid Finite-Difference Computation of Subsonic and Slight Supercritical Aerodynamic Flows," *AIAA Journal*, Vol. 13, May 1975, pp. 579-586.

[2] Shanks, D., "Nonlinear Transformations of Divergent and Slowly Convergent Sequences," *Journal of Mathematics and Physics*, Vol. 34, April 1955, pp. 1-42.

[3] Martin, E.D., "Progress in Application of Direct Elliptic Solvers to Transonic Flow Computations," in *Aerodynamic Analyses Requiring Advanced Computers*, Part II, NASA SP-347, 1975, pp. 839-870.

[4] Martin, E.D., "A Fast Semidirect Method for Computing Transonic Aerodynamic Flows," *AIAA 2nd Computational Fluid Dynamics Conference*, Hartford, Conn., June 1975, bound volume of papers. pp. 162-174, also to be published in *AIAA Journal*.

[5] Martin, E.D., "A Technique for Accelerating Iterative Convergence in Numerical Integration, with Application in Transonic Aerodynamics," *Padé Approximants Method and Its Application to Mechanics. Lecture Notes in Physics*, Vol. 47 (ed. by H. Cabannes), Springer-Verlag, Berlin, 1976, pp. 123-139.

[6] Aitken, A.C., "On Bernoulli's Numerical Solution of Algebraic Equations," *Proceedings of the Royal Society of Edinburgh*, Vol. 46, 1926, pp. 289-305.

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